LESSON

15 Optimization

Knowing the minimum and maximum points of a function is useful for graphing and even more useful in real-life situations. Businesses want to maximize their profits, builders want to minimize their costs, drivers want to minimize distances, and people want to get the most for their money. If we can represent a situation with a function, then the derivative will help find the optimal point.

If the derivative is zero or undefined at exactly one point, then this is very likely to be the optimal point. The *first derivative test* states that if the function increases before that point and decreases afterward, it is maximal (see Figure 15.1). Similarly, if the function decreases before the point and increases afterward, then the point is an absolute minimum.

The *second derivative test* states that if the second derivative is positive, then the function curves up, so a point of slope zero must be a minimum (see Figure 15.2). Similarly, if the second derivative is negative, the point of slope zero must be the highest point on the graph. Remember that we are assuming that only one point has slope zero or an undefined derivative.

If there are several points of slope zero and the function has a closed interval for a domain, then plug all the *critical points* (points of slope zero, points of undefined derivative, and the two endpoints of the interval) into the original function. The point with the highest y-value will be the absolute maximum, and the one with the smallest y-value will be the absolute minimum.

Figure 15.1

Figure 15.2

Example

A manager calculates that when x employees are working at the same time, the store makes a profit of $P(x) = 15x^2 - 48x - x^3$ dollars each hour. If there are ten employees and at least one must be working at any given time, how many employees should be scheduled to maximize profit?

Solution

This is an instance of a function on a closed interval because $1 \leq x \leq 10$ limits the options for x. The derivative of the profit function is $P'(x) = 30x - 48 - 3x^2$ which factors into $P'(x) = -3(x^2 - 10x + 16) = -3(x - 2)(x - 8)$. Thus, the derivative is zero at $x = 2$ and at $x = 8$.

Because more than two points have a slope of zero, we cannot use the first or second derivative tests. Instead, we evaluate each of our critical points. These are the points of slope zero, $x = 2$ and $x = 8$, plus the endpoints of our interval $x = 1$ and $x = 10$. These are evaluated as follows: $P(1) = -34$, $P(2) = -44$, $P(8) = 64$, and $P(10) = 20$. If the manager wants to maximize the store profit, eight employees should be scheduled at the time, because this will result in a maximal profit of \$64 each hour.

Example

A coffee shop owner calculates that if she sells cookies at \$$p$ each, she will sell $\frac{200}{p^2}$ cookies each day. If it costs her 20¢ to make each cookie, what price p will give her the greatest profit?

Solution

The function for profit is Profit = Revenue − Costs. If she charges \$$p$ per cookie, then she'll make and sell $\frac{200}{p^2}$ cookies each day. Thus, her revenue will be

$\left(\frac{200}{p^2}\right) \cdot p = \frac{200}{p}$ and her costs will be $\left(\frac{200}{p^2}\right) \cdot (0.20)$. Therefore, her profit function is $\text{Profit}(p) = \frac{200}{p} - \frac{40}{p^2}$. We limit this to $p > 0.20$ because the only optimal situation would be when the cookies were sold for more than it cost to make them.

The derivative is $\text{Profit}'(p) = -\frac{200}{p^2} + \frac{80}{p^3}$, which is zero when $\frac{80}{p^3} = \frac{200}{p^2}$ and therefore $80p^2 = 200p^3$, so either $p = 0$ or else $p = \frac{80}{200} = 0.40$. Because $p = 0$ is not in the domain, the only place where the derivative is zero is at $p = 40¢$.

Using the first derivative test, we see that $\text{Profit}'(0.30) = 740$ and $\text{Profit}'(0.50) = -160$, therefore our sign diagram for Profit' is as shown in Figure 15.3. So the absolute maximal profit occurs when the cookies are sold at 40¢.

Example

At \$1 per cup of coffee, a vendor sells 500 cups a day. When the price is increased to \$1.10, the vendor sells only 480 cups. If every 1¢ increase in price reduces the sales by two cups, what price per cup of coffee will maximize income?

Solution

Here, the income is Income = Price × Cups Sold. So if x = the number of pennies the price is increased, then $\text{Income}(x) = (1 + 0.01x) \cdot (500 - 2x)$. This simplifies to $\text{Income}(x) = 500 + 3x - 0.02x^2$. And, the derivative is $\text{Income}'(x) = 3 - 0.04x$. This is zero only when $x = \frac{3}{0.04} = 75$. The second derivative is $\text{Income}''(x) = -0.04$, which is negative, so $x = 75$ is maximal by the second derivative test. Thus, the maximal income will occur when the price is raised by $x = 75¢$ to \$1.75 per cup.

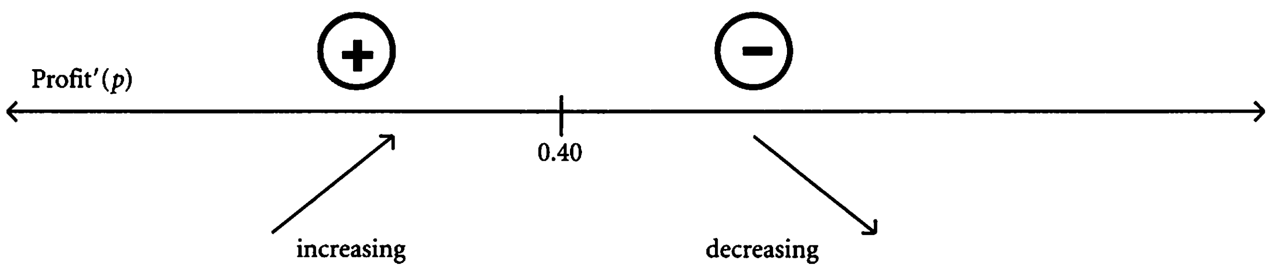


Figure 15.3

Example

A farmer wants to build a rectangular pen with 80 feet of fencing. The pen will be built against the side of a barn, so one side won't need a fence. What dimensions will maximize the area of the pen? See Figure 15.4.

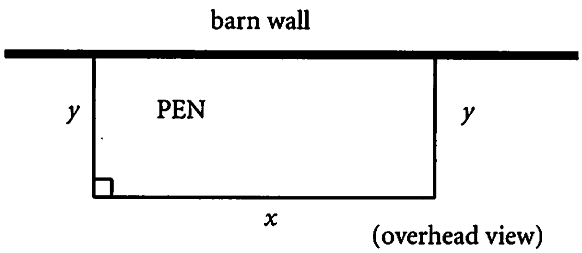


Figure 15.4

Solution

The area of the pen is Area $= x \cdot y$. We can't take the derivative of this just yet because there are two variables. We need to use the additional information regarding how much fencing exists; there are 80 feet of fencing. Because no fencing will be required against the barn wall, the total lengths of the fence will be $y + x + y = 80$, thus $x = 80 - 2y$. We can plug this into the formula for area in order to obtain Area $= x \cdot y = (80 - 2y) \cdot y$. Now we have a function of one variable Area$(y) = 80y - 2y^2$. The derivative is Area$'(y) = 80 - 4y$. This is zero only when $y = 20$. Using the second derivative test, Area$''(y) = -4$, thus the curve is concave down and the point $y = 20$ is the absolute maximum. The corresponding x-value is $x = 80 - 2y = 80 - 2(20) = 40$. Therefore, the pen with the maximal area will be $x = 40$ feet wide (along the barn) and $y = 20$ feet out from the barn wall.

Example

A manufacturer needs to design a crate with a square bottom and no top. It must hold exactly 32 cubic feet of shredded paper. What dimensions will minimize the material needed to make the crate (the surface area)? See Figure 15.5.

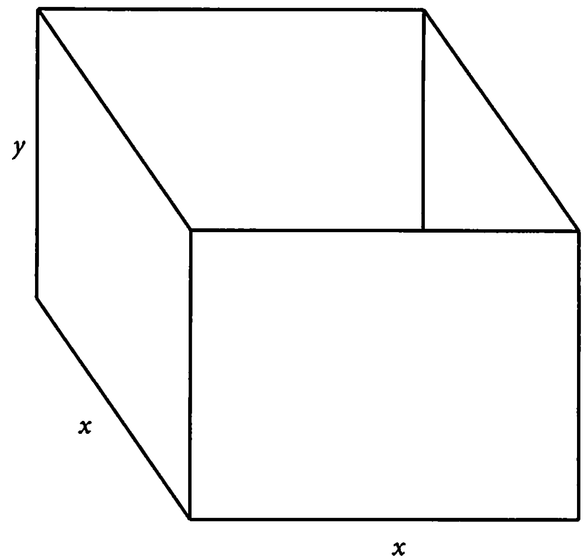


Figure 15.5

Solution

We want to minimize the surface area of the crate. The area of the box consists of four sides, each of area $x \cdot y$, plus the bottom, with an area of $x \cdot x = x^2$. Thus, the surface area is Area $= 4xy + x^2$. Again, we need to reduce this to a formula with only one variable in order to differentiate. We know that the volume must be 32 cubic feet, so Volume $= x^2y = 32$. Thus, $y = \frac{32}{x^2}$. When we plug this into the surface area function, we get:

$$\text{Area} = 4xy + x^2 = 4x\left(\frac{32}{x^2}\right) + x^2 = \frac{128}{x} + x^2.$$

The derivative is:

$$\text{Area}'(x) = -\frac{128}{x^2} + 2x,$$

which is zero when

$$-\frac{128}{x^2} + 2x = 0 \text{ or } x^3 = 64, \text{ so } x = 4.$$

The second derivative is:

$$\text{Area}''(x) = \frac{256}{x^3} + 2,$$

which is positive when $x = 4$. So the curve is concave up and the sole point of slope zero is the absolute minimum. Thus, the surface area of the crate will be minimized if $x = 4$ feet and $y = \frac{32}{x^2} = \frac{32}{4^2} = 2$ feet.

▶ Practice

1. Suppose a company makes a profit of $P(x) = \frac{1{,}000}{x} - \frac{5{,}000}{x^2} + 100$ dollars when it makes and sells $x > 0$ items. How many items should it make to maximize profit?

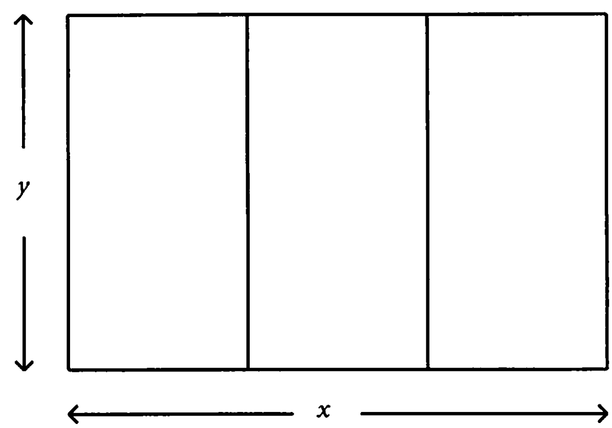
2. Suppose the profit of a company is $P(x) = 9x^2 + 40x - \frac{1}{3}x^3 + 1{,}000$ when it makes x items a day. What level of production will maximize profits?

3. When 30 orange trees are planted on an acre, each will produce 500 oranges a year. For every additional orange tree planted, each tree will produce 10 fewer oranges. How many trees should be planted to maximize the yield?

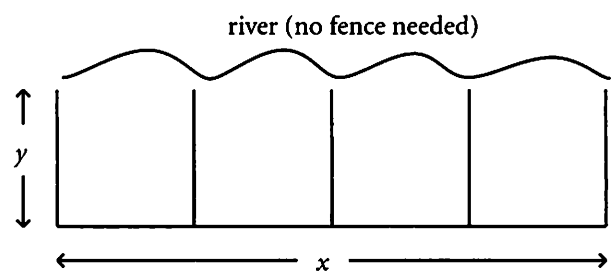
4. An artist can sell 20 copies of a painting at \$100 each, but for each additional copy she makes, the value of each painting will go down by a dollar. Thus, if 22 copies are made, each will sell for \$98. How many copies should she make to maximize her sales?

5. A garden has 200 pounds of watermelons growing in it. Every day, the total amount of watermelon increases by 5 pounds. At the same time, the price per pound of watermelon goes down by 1¢. If the current price is 90¢ per pound, how much longer should the watermelons grow in order to fetch the highest price possible?

6. A farmer has 400 feet of fencing to make three rectangular pens. What dimensions x and y will maximize the total area?



7. Four pens will be built along a river by using 150 feet of fencing. What dimensions will maximize the area of the pens?



8. A rectangular pen will be built using 100 feet of fencing. What dimensions will maximize the area?

9. The surface area of a can is $\text{Area} = 2\pi r^2 + 2\pi rh$, where the height is h and the radius is r. The volume is $\text{Volume} = \pi r^2 h$. What dimensions minimize the surface area of a can with volume 16π cubic inches?

10. A painter has enough paint to cover 600 square feet of area. What is the largest square-bottom box that could be painted (including the top, bottom, and all sides)?

11. A box with a square bottom will be built to contain 40,000 cubic feet of grain. The sides of the box cost 10¢ per square foot to build, the roof costs \$1 per square foot to build, and the bottom will cost \$7 per square foot to build. What dimensions will minimize the building costs?

12. A printed page will have a total area of 96 square inches. The top and bottom margins will be 1 inch each, and the left and right margins will be $1\frac{1}{2}$ inches each. What overall dimensions for the page will maximize the area of the space inside the margins?